Idealization of some weak separation axioms*

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Abstract

An ideal is a nonempty collection of subsets closed under heredity and finite additivity. The aim of this paper is to unify some weak separation properties via topological ideals. We concentrate our attention on the separation axioms between T_0 and $T_{\frac{1}{2}}$. We prove that if $(X, \tau, \mathcal{I})$ is a semi-Alexandroff $T_{\mathcal{I}}$-space and $\mathcal{I}$ is a τ-boundary, then $\mathcal{I}$ is completely codense.

1 Introduction

Regardless of the fact that for many years ideals had their significant impact in research in Topology it was probably the four articles of Hamlett and Janković [7, 8, 10, 11], which appeared almost a decade ago, that initiated the application of topological ideals in the generalization of most fundamental properties in General Topology.

Throughout the 90's several topological properties such as covering properties, connectedness, resolvability, extremal disconnectedness and submaximality have been generalized via topological ideals. Contributions in the field are due to (in alphabetical order) Abd El-Monsef, Ergun, Ganster, Hamlett, Janković, Lashien, Maki, Nasef, Noiri, Rose and Umehara.

Probably only separation axioms have been neglected from 'ideal point of view'. In this paper, we will consider only weak separation axioms – the ones between T_0 and $T_{\frac{1}{2}}$. Recall

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that a topological space (X,τ) is called a $T_{\frac{1}{2}}$*-space* if every singleton is open or closed. The importance of the separation axiom $T_{\frac{1}{2}}$ is probably given by the following (perhaps well-known) result: Every minimal $T_{\frac{1}{2}}$-space is compact and connected.

In Digital Topology [12] several spaces that fail to be T_1 are important in the study of the geometric and topological properties of digital images [12, 14]. Such is the case with the major building block of the digital n-space – the *digital line* or the so called *Khalimsky line*. This is the set of the integers, $\mathbb{Z}$, equipped with the topology $\mathcal{K}$, generated by $\mathcal{G}_{\mathcal{K}} = \{\{2n-1, 2n, 2n+1\} \colon n \in \mathbb{Z}\}$. Although the digital line is not a T_1-space, it satisfies the separation axiom $T_{\frac{1}{2}}$. This probably indicates that further knowledge (from more general point of view) of the behavior of topological spaces satisfying separation axioms below T_1 is required, which indeed is the intention of the present paper.

A *topological ideal* is a nonempty collection of subsets of a topological space (X,τ), which is closed under heredity and finite additivity. Except the trivial ideals, i.e. the minimal ideal $\{\emptyset\}$ and the maximal ideal $\mathcal{P}$, the following collections of sets form important ideals on any topological space (X,τ): the finite sets $\mathcal{F}$, the countable sets $\mathcal{C}$, the closed and discrete sets $\mathcal{CD}$, the nowhere dense sets $\mathcal{N}$, the meager sets $\mathcal{M}$, the scattered sets $\mathcal{S}$ (only when X is T_0), the bounded sets $\mathcal{B}$, the relatively compact sets $\mathcal{R}$, the hereditarily compact (resp. Lindelöf) sets $\mathcal{HK}$ (resp. $\mathcal{HL}$), the S-bounded sets $\mathcal{SB}$ [13] and (in the real line) the Lebesgue null sets $\mathcal{L}$.

An *ideal topological space* $(X,\tau,\mathcal{I})$ is a topological space (X,τ) and an ideal $\mathcal{I}$ on (X,τ). For an ideal topological space $(X,\tau,\mathcal{I})$ and a subset $A \subseteq X$, $A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ is called the *local function* of A with respect to $\mathcal{I}$ and τ [15] (where $\tau(x)$ is the open neighborhood filter at x). We simply write A^* instead of $A^*(\mathcal{I})$ in case there is no chance for confusion.

Note that $\mathrm{Cl}^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology $\tau^*(\mathcal{I})$ (also denoted by τ^* when there is no chance for confusion), finer than τ. A basis $\beta(\mathcal{I},\tau)$ for $\tau^*(\mathcal{I})$ can be described as follows: $\beta(\mathcal{I},\tau) = \{U \setminus I \colon U \in \tau \text{ and } I \in \mathcal{I}\}$. In general, β is not always a topology [10].

2 Weak separation axioms below T_1 via ideals

Definition 1 An ideal topological space $(X, \tau, \mathcal{I})$ is called a $T_{\mathcal{I}}$*-space* if for every subset $I \in \mathcal{I}$ of X and every $x \notin I$, there exists a set A_x containing x and disjoint from I such that A_x is open or closed.

Proposition 2.1 *If a topological space X is $T_{\mathcal{J}}$, then it is a $T_{\mathcal{I}}$-space for each ideal $\mathcal{I} \subseteq \mathcal{J}$.*

Proof. Let X be a $T_{\mathcal{J}}$-space. Suppose that $\mathcal{I}$ is an ideal with $\mathcal{I} \subseteq J$ and let $I \in \mathcal{I}$ and $x \notin I$. Then $I \in \mathcal{J}$ and there exists a set A_x which is open or closed such that $x \in A_x$ and $A_x \cap I = \emptyset$, so X is $T_{\mathcal{I}}$-space. $\square$

Observation 2.2 (i) Every $T_{\frac{1}{2}}$-space is a $T_{\mathcal{I}}$-space.

(ii) Every space is a $T_{\mathcal{CD}}$-space.

(iii) If $\mathcal{I}$ is an ideal on a set X, the topological space X with the family of closed sets $\mathcal{I} \cup \{X\}$, is a $T_{\mathcal{I}}$-space. Note that using the notation from the preceding section, this topology on X can be described as $\tau_T^*(\mathcal{I})$, where τ_T is the trivial topology in X, and in this case the basis $(\beta\mathcal{I}, \tau_T)$ is in fact the whole topology. On the other hand, if this topological space is a $T_{\mathcal{J}}$-space for another ideal $\mathcal{J}$, then the following relation between both ideals $\mathcal{I}$ and $\mathcal{J}$ are obtained: if $F \in \mathcal{J} \setminus \mathcal{I}$ then F is the intersection of some open sets of X. To see this, suppose that $F \in \mathcal{J} \setminus \mathcal{I}$, then for each $x \notin F$ there exists a closed set A_x disjoint from F that contains x (note that if $A_x = X \setminus I$, with $I \in \mathcal{I}$, is a non-empty open set disjoint from F, then $F \subseteq I$ and $F \in \mathcal{I}$). So, $F = \cap_{x \notin F}(X \setminus A_x)$ where $X \setminus A_x$ is open for each $x \notin F$.

In [16], Newcomb defined an ideal $\mathcal{I}$ on a space (X, τ) to be a τ*-boundary* if $\tau \cap \mathcal{I} = \{\emptyset\}$. A topological space (X, τ) is called *semi-Alexandroff* [2] if any intersection of open sets is semi-open, where a *semi-open* set is a set which can be placed between an open set and its closure. Complements of semi-open sets are called *semi-closed*.

In connection with our next result, we recall that a set A is called *locally dense* [4] or *preopen* if $A \subseteq \mathrm{Int}\overline{A}$. The collection of all preopen subsets of a topological space (X, τ) is denoted by $PO(X)$. An ideal $\mathcal{I}$ on a space $(X, \tau, \mathcal{I})$ is called *completely codense* [6] if

$PO(X) \cap \mathcal{I} = \{\emptyset\}$. Note that if $(\mathbb{R}, \tau)$ is the real line with the usual topology, then $\mathcal{C}$ is codense but not completely codense. It is shown in [6] that an ideal $\mathcal{I}$ is completely codense on (X, τ) if and only if $\mathcal{I} \subseteq \mathcal{N}$, i.e. if each member of $\mathcal{I}$ is nowhere dense.

Proposition 2.3 *If $(X, \tau, \mathcal{I})$ is a semi-Alexandroff $T_{\mathcal{I}}$-space and $\mathcal{I}$ is a τ-boundary, then $\mathcal{I}$ is completely codense.*

Proof. Let $I \in \mathcal{I}$. If $I = X$, i.e., if $\mathcal{I} = \mathcal{P}$, we have $X = \emptyset$, since $\mathcal{I}$ is a τ-boundary and we are done. Assume next that I is a proper subset of X. Since $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$-space, for every point $x \notin I$, we can find a set A_x such that $x \in A_x$, $A_x \cap I = \emptyset$ and A_x is open or closed. Set $U = \cup\{A_x \colon A_x \text{ is open}\}$ and $V = \cup\{A_x \colon A_x \text{ is not open}\}$. Clearly, $U \in \tau$ and V is semi-closed, since (X, τ) is a semi-Alexandroff space. Thus, $X \setminus I$ is the union of an open and a semi-closed set. Hence, I is the intersection of a semi-open and a semi-closed set or equivalently the union of an open set W and a nowhere dense set N (such sets are called simply-open). Note that $W = \emptyset$, since $\mathcal{I}$ is a τ-boundary and $W \in \mathcal{I}$ due to the heredity of the ideal $\mathcal{I}$. This shows that $I = N$, i.e., $\mathcal{I} \subseteq \mathcal{N}$. $\square$

Recall that a nonempty topological space (X, τ) is called *resolvable* [9] (resp. *$\mathcal{I}$-resolvable* [6]) if X is the disjoint union of two dense (resp. $\mathcal{I}$-dense) subsets, where a subset A of a topological space $(X, \tau, \mathcal{I})$ is called *$\mathcal{I}$-dense* [6] if every point of X is in the local function of A with respect to $\mathcal{I}$ and τ, i.e. if $A^*(\mathcal{I}) = X$.

Corollary 2.4 *If $(X, \tau, \mathcal{I})$ is a semi-Alexandroff $T_{\mathcal{I}}$-space such that $\mathcal{I}$ is a τ-boundary, then resolvability of (X, τ) implies automatically the $\mathcal{I}$-resolvability of $(X, \tau, \mathcal{I})$.*

Several already defined separation axioms are either implied or equivalent to the 'ideal separation axiom' in Definition 1 when a certain ideal is considered.

Recall that a topological space (X, τ) is called a *$T_{\frac{1}{4}}$-space* [1] (resp. a *$T_{\frac{1}{3}}$-space* [3]) if for every finite (resp. compact) subset S of X and every $x \notin S$ there exists a set S_x containing F and disjoint from $\{x\}$ such that A_x is open or closed.

Proposition 2.5 *Let (X, τ) be a topological space. Then the following conditions are valid:*

(i) (X, τ) *is a* $T_{\frac{1}{4}}$*-space if and only if* $(X, \tau, \mathcal{I})$ *is a* $T_{\mathcal{F}}$*-space.*

(ii) *Every* $T_{\frac{1}{3}}$*-space is a* $T_{\mathcal{HK}}$*-space.*

(iii) (X, τ) *is a* $T_{\frac{1}{2}}$*-space if and only if* $(X, \tau, \mathcal{I})$ *is a* $T_{\mathcal{P}}$*-space.*

(iv) *Every* $T_{\mathcal{I}(K)}$*-space is a* $T_{\frac{1}{3}}$*-space, where* $\mathcal{I}(K)$ *is the ideal formed by the subsets of the compact sets of the topological space.*

Proof. (i) and (ii) follow directly from the definitions.

(iii) Assume that (X, τ) is a two-space, let P be an arbitrary subset of X and let $x \notin P$. Then $\{x\}$ is open or closed, which points out that $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{P}}$-space. Conversely, if $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{P}}$-space and $x \in X$, then there exists a set A_x, which is open or closed such that $x \in A_x$ and $A_x \cap X \setminus \{x\} = \emptyset$. Clearly, $\{x\} = A_x$. This shows that every singleton of (X, τ) is open or closed or equivalently (X, τ) is a $T_{\frac{1}{2}}$-space.

(iv) Let X be a $T_{\mathcal{I}(K)}$-space, K a compact subset of X and $x \notin K$. Then $K \in \mathcal{I}(K)$, so there exists a subset A_x of X which is either open or closed such that $x \in A_x$ and $A_x \cap K = \emptyset$, so X is a $T_{\frac{1}{3}}$-space. $\square$

Example 2.6 We give an example of a $T_{\mathcal{HK}}$-space which is not a $T_{\frac{1}{3}}$-space. We topologize the real line $\mathbb{R}$ by declaring the following non-trivial open sets:

(1) all singletons except $\{1\}$ and $\{2\}$;

(2) all cofinite sets containing 2 but not 1;

(3) all cofinite sets containing the set $\{1, 2\}$.

We show that this is a $T_{\mathcal{HK}}$-space. Let $A \subseteq \mathbb{R}$ be hereditarily compact and let $x \notin A$. If $x \neq 2$, then $\{x\}$ is open or closed and we are done. If $x = 2$, then $1 \notin A$, in which case A is open, or $1 \in A$, in which case $A \setminus \{1\}$ is finite for A is hereditarily compact and hence A is closed.

Next we observe that the space is not a $T_{\frac{1}{3}}$-space. For note that $A \setminus \{2\}$ is compact but $\{2\}$ is neither open nor closed.

If X is a set, a *subideal* (or an *h-family*) $\mathcal{I}$ on X, is a family of subsets of X which is closed under the subset relation. A family which is closed under finite additivity is called an *FA-family.* Sometimes, in the definition of $T_{\mathcal{I}}$-spaces, we might not require that the family $\mathcal{I}$ is an ideal, we take the more general approach of considering subideals and FA-families. The FA-family formed by Lindelöf subsets of a given topological space will be denoted by $\mathcal{I}(L)$.

A set $A \subseteq (X, \tau)$ is called *discretely finite* (= df-set) (resp. *discretely countable* (=dc-set)) [6] if for every point $x \in A$, there exists $U \in \tau$ containing x such that $U \cap A$ is finite (resp. countable). The subideals of all df-sets (resp. dc-sets) will be denoted by $\mathcal{DF}$ (resp. $\mathcal{DC}$).

Using Proposition 2.1, which is valid for pairs of subideals and pairs of FA-families, we obtain the following diagram where the relations between different classes of $T_{\mathcal{I}}$-spaces are shown.

$$\begin{tikzcd}
T_1 \arrow[r] \arrow[d] & T_{\mathcal{HL}} \arrow[r] \arrow[d] & T_{\mathcal{I}(L)} \arrow[r] \arrow[d] & T_{\mathcal{C}} \arrow[r] \arrow[d] & T_{\mathcal{DC}} \arrow[r] \arrow[d] & T_0 \arrow[d] \\
T_{\mathcal{P}}=T_{\frac{1}{2}} \arrow[r] & T_{\mathcal{HK}} \arrow[r] & T_{\mathcal{I}(K)} \arrow[r] & T_{\mathcal{F}} \arrow[r] & T_{\mathcal{DF}} \arrow[r] \arrow[ur] & T_{\emptyset}
\end{tikzcd}$$

Example 2.7 A $T_{\frac{1}{4}}$-space which is not a $T_{\mathcal{HK}}$-space. The topological space in [3], Example 3.1 is a $T_{\frac{1}{4}}$-space. This is the set X of non-negative integers with the topology whose open sets are those which contain 0 and have finite complement (so closed sets are the finite sets that do not contain 0). Each subset of X is compact, then $\mathcal{H}K = \mathcal{P}$ and X is not a $T_{\mathcal{HK}}$-space because it is not a $T_{\frac{1}{2}}$-space.

Recall that an *α-space* (or a *nodec space*) is a space where $\tau^*(\mathcal{N}) = \tau$. Note that α-spaces need not be even T_0.

Remark 2.8 1) Every α-space (X, τ) is a $T_{\mathcal{N}}$-space: Let $N \in \mathcal{N}$ and $x \notin N$. Since X is an α-space, then every nowhere dense subset is closed (and discrete). Thus N is closed, which shows that X is a $T_{\mathcal{N}}$-space.

(2) It is not difficult to find an example of a $T_{\mathcal{N}}$-space which is not an α-space. Note that even a metric space (for example the real line with usual topology) need not be an α-space.

A subset S of a space $(X, \tau, \mathcal{I})$ is a topological space with an ideal $\mathcal{I}_S = \{I \in \mathcal{I}\colon I \subseteq S\} = \{I \cap S\colon I \in \mathcal{I}\}$ on S [5].

Proposition 2.9 *Every subspace of a $T_{\mathcal{I}}$-space is a $T_{\mathcal{I}_A}$-space.*

Proof. Let $(X, \tau, \mathcal{I})$ be a $T_{\mathcal{I}}$-space and let $A \subseteq X$. If $J \in \mathcal{I}_A$ is a subset of $(A, \tau|A, \mathcal{I}_A)$ and $x \in A \setminus J$, then there exists $I \in \mathcal{I}$ such that $I \cap A = J$. Since $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$-space, there exists a set $B \supseteq I$ such that either $B \in \tau$ or $X \setminus B \in \tau$ and $x \notin B$. Clearly, $B \cap A$ is open or closed in the subspace $(X, \tau|A)$. This shows that $(A, \tau|A, \mathcal{I}_A)$ is a $T_{\mathcal{I}_A}$-space. $\square$

Proposition 2.10 *Every topology finer than a $T_{\mathcal{HK}}$-topology is a $T_{\mathcal{HK}}$-topology.*

Proof. Let $\tau^* \subseteq \tau$ be two topologies on a set X. Then, $\mathcal{HK}_\tau \subseteq \mathcal{HK}_{\tau^*}$: let $H \in \mathcal{HK}_\tau$ be and $F \subseteq H$, then F is τ-compact, so it is τ^*-compact and $H \in \mathcal{HK}_{\tau^*}$.

Suppose that (X, τ^*) is a $T_{\mathcal{HK}_{\tau^*}}$-space. Then (X, τ) is a $T_{\mathcal{HK}_{\tau^*}}$-space: let $H \in \mathcal{HK}_{\tau^*}$ and $x \notin H$, then there exists A_x which is τ^*-closed or τ^*-open such that $x \notin A_x$ and $A_x \cap H = \emptyset$. Then A_x is τ-closed or τ-open and so (X, τ) is a $T_{\mathcal{HK}_{\tau^*}}$-space. Now, using that $\mathcal{HK}_\tau \subseteq \mathcal{HK}_{\tau^*}$ and Proposition 2.1, we obtain that (X, τ) is a $T_{\mathcal{HK}_\tau}$-space. $\square$

Remark 2.11 Note that if $\mathcal{I}$ is an ideal of a topological space X that does not depend on the topology, for example $\mathcal{F}$ or $\mathcal{C}$, then with a proof similar to the one below every topology in X finer that a $T_{\mathcal{I}}$-topology is a $T_{\mathcal{I}}$-topology.

A property $\mathcal{P}$ of an ideal topological space is called *I-topological* if $(Y, \sigma, \mathcal{J})$ has the property $\mathcal{P}$ whenever $(X, \tau, \mathcal{I})$ has the property $\mathcal{P}$ and $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is an $\mathcal{I}\mathcal{J}$-homeomorphism, i.e., $f\colon (X, \tau) \to (Y, \sigma)$ is a homeomorphism and $f(\mathcal{I}) = \mathcal{J}$. It is well-known that the property $T_{\frac{1}{2}}$ is topological. The following proposition is an ideal generalization of that result.

Proposition 2.12 *The property $T_{\mathcal{I}}$ is I-topological.*

Proof. Assume that $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$-space and that $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is an $\mathcal{I}\mathcal{J}$-homeomorphism. Let $J \in \mathcal{J}$ and let $y \in Y$ such that $y \notin J$. Since $f\colon (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is an I-homeomorphism, then there exists $x \in X$ and $I \in \mathcal{I}$ such that $f(I) = J$, $y = f(x)$ and $x \notin I$. Then, find a set $A_x \subseteq X$ such that $x \in A_x$, $A_x \cap I = \emptyset$ and A_x is open or closed (such choice is possible for $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$-space). Since f is open and closed, then $f(A_x)$ is open or closed subset of (Y, σ). Moreover, $y \in f(A_x)$ and $f(A_x) \cap J = \emptyset$. This shows that $(Y, \sigma, \mathcal{J})$ is a $T_{\mathcal{J}}$-space. □

Question. How could we ideally extend the following result: Every minimal $T_{\frac{1}{2}}$-space is compact and connected.

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